Lösung (13. Übung)

1. (i) 2 (ii) $\frac{\pi}{3}$ (iii) 2π (iv) π (v) divergent

2. $y = \frac{1}{x}$, $x \geq 1$

$$V_x = \pi \int_1^\infty \frac{1}{x^2}\,dx = \pi \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = -\pi \lim_{b\to\infty}\left(\frac{1}{b} - 1\right) = \pi$$

$$A_x = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi \lim_{b\to\infty} \int_1^b \frac{\sqrt{1+x^4}}{x^3}\,dx =$$

$$\int \frac{\sqrt{1+x^4}}{x^3}\,dx = \frac{1}{2}\int \frac{u^2\,du}{\sqrt[4]{u^2-1}^{\,3}} = \frac{1}{2}\int \frac{u^2\,du}{(u^2-1)\sqrt{u^2-1}} = \frac{1}{2}\left[\ln\left(u+\sqrt{u^2-1}\right) - \frac{u}{\sqrt{u^2-1}}\right] \quad (*)$$

$\sqrt{1+x^4} = u$

$x^4 = u^2-1 \quad x = \sqrt[4]{u^2-1}$

$4x^3\,dx = 2u\,du$

$$= \lim_{b\to\infty} \pi\left\{\ln\left(\sqrt{1+b^4} - b^2\right) - \ln\left(\sqrt{2}+1\right) - \frac{\sqrt{1+b^4}}{b^2} + \sqrt{2}\right\} \text{ ex. nicht.}$$

Die Farbe reicht nicht.

3. (i) $\displaystyle\int_a^\infty \frac{dx}{x^s} = \lim_{b\to\infty}\int_a^b \frac{dx}{x^s} = \lim_{b\to\infty}\left(\frac{1}{(1-s)x^{s-1}}\Big|_a^b\right) = \lim_{b\to\infty}\left(\frac{1}{(1-s)b^{s-1}} - \frac{1}{(1-s)a^{s-1}}\right) = \begin{cases} \frac{1}{(s-1)a^{s-1}}, & s>1 \\ \text{ex. nicht}, & s\leq 1\end{cases}$

$a>0$

(ii) $\displaystyle\int_0^\infty \frac{dx}{x^s} = \lim_{a\to 0}\int_a^c \frac{dx}{x^s} + \lim_{b\to\infty}\int_c^b \frac{dx}{x^s} = \lim_{a\to 0}\frac{1}{(s-1)a^{s-1}} + \lim_{b\to\infty}\frac{1}{(1-s)b^{s-1}}$ ex. nicht, $\forall s\in\mathbb{R}$

$a<c<b$

(iii) $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\arctan b = \frac{\pi}{2}$

(iv) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon_1\to 0}\int_{-1+\varepsilon_1}^0 \frac{dx}{\sqrt{1-x^2}} + \lim_{\varepsilon_2\to 0}\int_0^{1-\varepsilon_2}\frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon_1\to 0}\arcsin(-1+\varepsilon_1) + \lim_{\varepsilon_2\to 0}\arcsin(1-\varepsilon_2) = \pi$

4. (i) $\displaystyle\int_0^3 \frac{dx}{x-2}$ ex. nicht CHW: $\displaystyle\lim_{\varepsilon\to 0}\left[\int_0^{2-\varepsilon}\cdots + \int_{2+\varepsilon}^3 \cdots\right] = -\ln 2$

(ii) ex. nicht CHW: ex. nicht

(iii) $2+2\sqrt{2}$ CHW: $2+2\sqrt{2}$

(iv) $\displaystyle\int_{-1}^1 \frac{dx}{x} = \lim_{\varepsilon_1\to 0}\int_{-1}^{\varepsilon_1}\frac{dx}{x} + \lim_{\varepsilon_2\to 0}\int_{\varepsilon_2}^1 \frac{dx}{x}$ = ex. nicht. CHW: $\displaystyle\int_{-1}^1 \cdots = \lim_{\varepsilon\to 0}\left[\int_{-1}^{\varepsilon} \cdots + \int_{\varepsilon}^1 \cdots\right] = 0$

(v) $\displaystyle\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right|$ ex. nicht, CHW: 0

5. a)

$$J_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$$

$$J_1 = \frac{1}{a}\arctan\frac{x}{a}$$

$$J_2 = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\arctan\frac{x}{a}$$

b) $\int_0^\infty \frac{dx}{(x^2+a^2)^n}$ für $n=1$ ist $J_1 = \frac{\pi}{2a}$

$n=2$ ist J_2: $\frac{\pi}{4a^3}$

ist J_n: $\frac{3\cdot5\cdot7\cdot\ldots\cdot(2n-3)}{a^{n+1}2^n(n-1)!}\pi$

6. $\Gamma(n) = (n-1)!$; $n=1,2,\ldots$

7. a) $\int_0^\infty x^\gamma e^{-x}dx = \lim_{b\to\infty}\int_0^b x^\gamma e^{-x}dx$ ex. für $\gamma \geq 0$.

$u = x^\gamma \quad v' = e^{-x}$

$u' = \gamma x^{\gamma-1} \quad v = -e^{-x}$

für $\int_1^\infty x^\gamma e^{-x}dx$ gilt: $\lim_{x\to\infty} x^{\gamma+2}e^{-x} = 0$, $\gamma\in\mathbb{R}$

also existiert eine Konstante M für alle $x \geq 1$

$$x^\gamma e^{-x} \leq \frac{M}{x^2}$$

Aus der Existenz von $\int_1^\infty \frac{dx}{x^2}$ folgt die Existenz von $\int_1^\infty x^\gamma e^{-x}dx$ für jedes $\gamma\in\mathbb{R}$

b) (i) Es sei $\beta > \alpha+1$. Dann gilt für $x \geq 1$

$$\frac{x^\alpha}{1+x^\beta} \leq x^{\alpha-\beta}$$

Wegen $\beta-\alpha > 1$ existiert $\int_1^\infty x^{\alpha-\beta}dx$ und damit $\int_1^\infty \frac{x^\alpha}{1+x^\beta}dx$

(ii) Es sei $\beta \leq \alpha+1$. Ist $\alpha < -1$ so gilt $\frac{x^\alpha}{1+x^\beta} \leq x^\alpha \Rightarrow$ Integral ex.

$\alpha = -1$: $\frac{x^\alpha}{1+x^\beta} \geq \frac{1}{2x} \Rightarrow$ Integral divergiert

$\alpha > -1$ so gilt

$$\int_1^B \frac{x^\alpha}{1+x^\beta}dx \geq \int_1^B \frac{x^\alpha}{1+x^{\alpha+1}}dx = \frac{1}{\alpha+1}\ln\frac{1+B^{\alpha+1}}{2} \Rightarrow \text{Integral divergiert.}$$

Hausaufgaben:

1. $J = \pi - 2$

$$\lim_{\varepsilon_1 \to 0} \underbrace{\int_{-1+\varepsilon_1}^{0} \frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx}_{\Downarrow} + \lim_{\varepsilon_2 \to 0} \int_{0}^{1-\varepsilon_2} \frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx \quad \text{existieren}$$

$$\int \left(\frac{1}{\sqrt{1-x^2}} - 1\right) dx$$

$$= \arcsin x - x$$

2. NST des Nenners $x_1 = 0$, $x_2 = -2$, $x_3 = 3$ kritische Stelle $x = 0$

$$\frac{1}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

$$1 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$$

$x = 0$: $A = -\frac{1}{6}$

$x = 3$: $B = \frac{1}{15}$

$x = -2$: $C = \frac{1}{10}$

$$\int \cdots = -\frac{1}{6}\ln|x| + \frac{1}{15}\ln|x-3| + \frac{1}{10}\ln|x+2|$$

$$= \ln \frac{|x-3|^{\frac{1}{15}}\,|x+2|^{\frac{1}{10}}}{|x|^{\frac{1}{6}}}$$

$\int_{-1}^{1} \frac{dx}{x(6+x-x^2)}$ existiert nicht als uneigentl. Integral

$$\left(\lim_{\varepsilon_1 \to 0} \int_{-1}^{\varepsilon_1} \cdots + \lim_{\varepsilon_2 \to 0} \int_{\varepsilon_2}^{1} \cdots \right)$$

CHW: $\lim_{\varepsilon \to 0} \left[\int_{-1}^{\varepsilon} \cdots + \int_{\varepsilon}^{1} \cdots \right] = \lim_{\varepsilon \to 0} \Big[\cancel{-\frac{1}{6}\ln|\varepsilon|} + \cancel{\frac{1}{15}\ln|\varepsilon-3|} + \cancel{\frac{1}{10}\ln|\varepsilon+2|} - \frac{1}{15}\ln 4 + \frac{1}{15}\ln 2 + \frac{1}{10}\ln 3 + \cancel{\frac{1}{6}\ln\varepsilon} - \cancel{\frac{1}{15}\ln|\varepsilon-3|} - \cancel{\frac{1}{10}\ln|\varepsilon+2|} \Big] = \underline{-\frac{1}{15}\ln 2 + \frac{1}{10}\ln 3}$